



# Markov Processes with Restart

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**RESEARCH  
REPORT**

**N° 8000**

June 2012

Project-Team Maestro





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Research Report n° 8000 — June 2012 — 12 pages

**Abstract:** We consider a general honest homogeneous continuous-time Markov process with restarts. The process is forced to restart from a given distribution at time moments generated by an independent Poisson process. The motivation to study such processes comes from modeling human and animal mobility patterns, restart processes in communication protocols, and from application of restarting random walks in information retrieval. We provide a connection between the transition probability functions of the original Markov process and the modified process with restarts. We give closed-form expressions for the invariant probability measure of the modified process. When the process evolves on the Euclidean space there is also a closed-form expression for the moments of the modified process. We show that the modified process is always positive Harris recurrent and exponentially ergodic with the index equal to (or bigger than) the rate of restarts. Finally, we illustrate the general results by the standard and geometric Brownian motions.

**Key-words:** Markov Processes with Restart, Positive Harris Recurrence, Exponential Ergodicity, Standard and Geometric Brownian Motions

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## Les Processus de Markov avec Redémarrage

**Résumé :** Nous considérons un processus de Markov en temps continu qui est général, honnête et homogène. Le processus est forcé à redémarrer à partir d'une distribution donnée à des moments de temps générés par un processus indépendant de Poisson. La motivation pour étudier ce type de processus vient de la modélisation de la mobilité humaine et animale, des procédures avec le redémarrage dans les protocoles de communication et de l'application des marches aléatoires avec le redémarrage en recherche d'information. Nous fournissons une connexion entre les fonctions de probabilité de transition du processus de Markov originale et du processus modifié avec le redémarrage. Nous donnons des expressions explicites de la mesure invariante du processus modifié. Lorsque le processus évolue dans l'espace Euclidien, il y a également une expression explicite pour les moments du processus modifié. Nous montrons que le processus modifié est à la fois Harris positif récurrent et ergodique exponentiel avec l'indice égal à (ou plus grand que) le taux de redémarrage. Enfin, nous illustrons les résultats généraux par les cas classiques de mouvements Browniens standard et géométrique.

**Mots-clés :** Processus de Markov avec Redémarrage, Positive Harris Recurrence, Ergodicité Exponentiel, Mouvements Browniens Standard et Géométrique

## 1 Introduction

Many phenomena in nature and technology can be modeled by Markov processes which restart from time to time. Human and animal movements can be modeled by Markov processes that restart from some locations [8, 19]. Facing the congestion of the Internet traffic, the Internet users tend to restart their sessions [11, 14] or a protocol governing the rate of information transmission invokes restart routines (i.e., Slow-Start routine in the Internet Transmission Control Protocol) [18]. The celebrated PageRank algorithm [3] in information retrieval models the behavior of a random surfer, who decides to restart web surfing from time to time. The heat kernel PageRank [4] is a continuous-time analog of the original discrete-time algorithm. The restart policy is also used to speedup the Las Vegas type randomized algorithms [1, 13].

Motivated by the above mentioned phenomena, we study a general honest homogeneous continuous-time Markov process with restarts. We assume that the process of restarts is modeled by an independent Poisson process. In the next Section 2 we derive a formula which makes a connection between the transition probability functions of the original Markov process and the Markov process modified by restarts. As a corollary we obtain a closed-form expression for the invariant probability measure of the modified process. We also show that the modified process is always positive Harris recurrent and exponentially ergodic with the index equal to (or bigger than) the rate of restarts. In Section 3 we provide bounds for the moments of the modified process and investigate the limiting behaviour of the modified process when the rate of restarts goes to zero. We conclude the paper with Section 4 where we consider the restart modifications of the two classical Markov processes: the standard Brownian motion with drift and the geometric Brownian motion. It is very curious to observe that in the case of the geometric Brownian motion the number of finite moments of the modified process depends on the rate of restarts. Thus, even a small change in the value of the restart rate can lead to dramatic changes of the properties of the geometric Brownian motion with restarts.

## 2 Main results

In accordance with [12], we consider a homogeneous continuous-time Markov process  $\{X(t) : t \in [0, \infty)\}$  in a Borel space  $(E, \mathcal{E})$  characterized by the initial distribution  $\gamma(\cdot)$ , and the honest transition (probability) function  $P(t, x, dy)$ , satisfying the following properties:

1.  $P(t, x, \cdot)$  is a probability measure on  $E$ ,
2.  $P(0, x, \Gamma) = 1\{x \in \Gamma\}$ ,
3. for each fixed  $\Gamma \in \mathcal{E}$ ,  $t \in [0, \infty)$ ,  $P(t, x, \Gamma)$  is a jointly measurable function with respect to  $(t, x) \in [0, \infty) \times E$ ,

4. and the Chapman-Kolmogorov equation takes place

$$P(t + s, x, \Gamma) = \int_E P(s, y, \Gamma) P(t, x, dy).$$

Throughout this paper, all the processes are from the same probability space  $(\Omega, \mathcal{F}, P)$ . We further use the notations  $P_x, E_x, Var_x$  when the initial state of the concerned process is  $x \in E$ .

The goal of the present work is to analyse a modification of the Markov process  $\{X(t) : t \in [0, \infty)\}$  introduced in the above by forcing the process to restart with a given restart distribution  $\nu(\Gamma)$  after an exponentially distributed random time, i.e., the process counting restarts represents a standard Poisson process with the rate, say,  $\lambda > 0$ , independent of  $\{X(t) : t \in [0, \infty)\}$ . In the following theorem we characterize the transition function of the modified Markov process, which is denoted as  $\{\tilde{X}(t) : t \in [0, \infty)\}$ .

**Theorem 1** *Let  $P(t, x, \Gamma)$  be the transition function of the Markov process  $\{X(t) : t \in [0, \infty)\}$  taking values in a Borel space  $(E, \mathcal{E})$ . Then, the modified Markov process  $\{\tilde{X}(t), t \in [0, \infty)\}$  that restarts from a distribution  $\nu$  after (independent) exponentially distributed random times with mean  $1/\lambda$  has the following transition function*

$$\tilde{P}_\nu(t, x, \Gamma) = e^{-\lambda t} P(t, x, \Gamma) + \int_E \int_0^t \lambda e^{-\lambda s} P(s, y, \Gamma) ds \nu(dy). \quad (1)$$

**Proof:** Denote by  $N(t)$  the number of restarts up to time  $t$  and by  $S$  the time elapsed since the time moment of the last restart before  $t$ . Then, we have

$$\begin{aligned} \tilde{P}_\nu(t, x, \Gamma) &= P[\{\tilde{X}(t) \in \Gamma\} \cap \{N(t) = 0\} | \tilde{X}(0) = x] \\ &\quad + P[\{\tilde{X}(t) \in \Gamma\} \cap \{N(t) > 0\} | \tilde{X}(0) = x]. \end{aligned} \quad (2)$$

The first term in the above equation is given by

$$P[\{\tilde{X}(t) \in \Gamma\} \cap \{N(t) = 0\} | \tilde{X}(0) = x] = e^{-\lambda t} P(t, x, \Gamma).$$

Let us now calculate the second term in (2). Note that the second term in (2) can be expressed as

$$P[\{\tilde{X}(t) \in \Gamma\} \cap \{N(t) > 0\} | \tilde{X}(0) = x] = \int_E \int_0^t P(s, y, \Gamma) dF(s) \nu(dy), \quad (3)$$

where  $F(s) = P[S \leq s | N(t) > 0]$ . Let  $T_k$  denote the  $k$ -th restart moment from time zero. If  $N(t) = n > 0$ , then the restart times  $T_1, \dots, T_n$  have the same distribution as the order statistics corresponding to  $n$  independent random variables uniformly distributed on the interval  $(0, t)$  [17], and therefore,

$$P[T_n \leq \tau | N(t) = n] = \left(\frac{\tau}{t}\right)^n, \quad \tau \in [0, t], \quad n = 1, 2, \dots$$

Then,

$$P[S \leq s | N(t) = n] = P[T_n > t - s | N(t) = n] = 1 - \left(\frac{t-s}{t}\right)^n, \quad n = 1, 2, \dots,$$

and, consequently,

$$F(s) = \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \left[ 1 - \left(\frac{t-s}{t}\right)^n \right] = e^{-\lambda t} [e^{\lambda t} - e^{\lambda(t-s)}] = 1 - e^{-\lambda s}.$$

Hence,

$$P[\{\tilde{X}(t) \in \Gamma\} \cap \{N(t) > 0\} | \tilde{X}(0) = x] = \int_E \int_0^t \lambda e^{-\lambda s} P(s, y, \Gamma) ds \nu(dy),$$

and the formula (1) follows.  $\square$

**Corollary 1** *The measure*

$$q_\nu(\Gamma) = \int_E \int_0^\infty \lambda e^{-\lambda s} P(s, y, \Gamma) ds \nu(dy) \quad (4)$$

*is an invariant probability measure for  $\tilde{P}_\nu(t, x, \Gamma)$ .*

**Proof:** Since  $P(s, y, \cdot)$  is a probability measure for all  $s$  and  $y$ , by the Vitali-Hahn-Saks theorem  $q_\nu(\cdot)$  is a probability measure as well. Let us show that it is indeed an invariant measure. It is enough to show this for  $\nu(dy) = 1\{w \in dy\}$ , where  $w \in E$  is fixed, and in that case we also denote  $q_\nu$  by  $q_w$  and  $\tilde{P}_\nu$  by  $\tilde{P}_w$  for brevity. By Theorem 1, we obtain

$$\begin{aligned} \int_E q_w(dz) \tilde{P}_w(t, z, \Gamma) &= \int_E q_w(dz) \left[ e^{-\lambda t} P(t, z, \Gamma) + \int_0^t \lambda e^{-\lambda s} P(s, w, \Gamma) ds \right] \\ &= e^{-\lambda t} \int_E q_w(dz) P(t, z, \Gamma) + \int_0^t \lambda e^{-\lambda s} P(s, w, \Gamma) ds. \end{aligned}$$

Substituting into the above equation

$$q_w(\Gamma) = \int_0^\infty \lambda e^{-\lambda s} P(s, w, \Gamma) ds$$

and using the Chapman-Kolmogorov equation yields

$$\begin{aligned} \int_E q_w(dz) \tilde{P}_w(t, z, \Gamma) &= e^{-\lambda t} \int_0^\infty \lambda e^{-\lambda s} P(s+t, w, \Gamma) ds + \int_0^t \lambda e^{-\lambda s} P(s, w, \Gamma) ds \\ &= \int_t^\infty \lambda e^{-\lambda s'} P(s', w, \Gamma) ds' + \int_0^t \lambda e^{-\lambda s} P(s, w, \Gamma) ds \\ &= \int_0^\infty \lambda e^{-\lambda s} P(s, w, \Gamma) ds = q_w(\Gamma), \end{aligned}$$

which concludes the proof of the corollary.  $\square$

**Remark 1** We note that alternatively formula (4) can be rewritten as

$$q_\nu(\Gamma) = \int_E \lambda R(y, \Gamma) \nu(dy),$$

where  $R(y, \Gamma) = \int_0^\infty e^{-\lambda s} P(s, y, \Gamma) ds$  is the resolvent operator.

Now we are ready to prove that the modified process is positive Harris recurrent and exponentially ergodic with index  $\lambda$ . Before this, let us remind the definitions.

Recall that a homogeneous continuous-time Markov process  $\{\tilde{X}(t), t \in [0, \infty)\}$  is called Harris recurrent if there exists a non-trivial  $\sigma$ -finite (recurrence) measure  $\mu$  on  $\mathcal{E}$  such that for each  $x \in E$ ,

$$\tau_\Gamma := \inf\{t \geq 0 : \tilde{X}(t) \in \Gamma\} < \infty, \quad P_x - \text{a.s.},$$

whenever  $\mu(\Gamma) > 0$ , see [7, 15].

Then, if a Harris recurrent process admits an invariant probability measure, it must be unique, and the process is further called positive Harris recurrent, see [7, 15].

A homogeneous continuous-time Markov process  $\{\tilde{X}(t), t \in [0, \infty)\}$  is called exponentially ergodic with index  $\alpha$  if there exist a probability measure  $\mu(\cdot)$  on  $\mathcal{E}$ , a finite valued function  $M(\cdot)$  on  $E$ , and a constant  $\alpha > 0$ , satisfying  $\|\tilde{P}(t, x, \cdot) - \mu(\cdot)\|_{TV} \leq M(x)e^{-\alpha t}$  for every  $x \in E$ , where  $\|\cdot\|_{TV}$  denotes the total variation norm, see [5].

**Theorem 2** *The modified Markov process  $\{\tilde{X}(t), t \in [0, \infty)\}$  is positive Harris recurrent and exponentially ergodic with index equal to (or bigger than) the rate of restarts  $\lambda$ , and the following inequality takes place*

$$|q_\nu(\Gamma) - \tilde{P}_\nu(t, x, \Gamma)| \leq e^{-\lambda t}, \quad \forall x \in E, \forall \Gamma \in \mathcal{E}. \quad (5)$$

**Proof:** The modified process  $\{\tilde{X}(t), t \in [0, \infty)\}$  under consideration is Harris recurrent with the recurrence measure given by the restart distribution  $\mu(\cdot) = \nu(\cdot)$ . From Corollary 1 we conclude that  $\{\tilde{X}(t), t \in [0, \infty)\}$  is positive Harris recurrent with the unique invariant probability measure given by (4). Let us now prove (5). We see

$$\begin{aligned} |q_\nu(\Gamma) - \tilde{P}_\nu(t, x, \Gamma)| &= \left| \int_E \int_t^\infty \lambda e^{-\lambda s} P(s, y, \Gamma) ds \nu(dy) - e^{-\lambda t} P(t, x, \Gamma) \right| \\ &= \left| \int_E \int_t^\infty \lambda e^{-\lambda s} P(s, y, \Gamma) ds \nu(dy) - \int_t^\infty \lambda e^{-\lambda s} P(t, x, \Gamma) ds \right| \\ &= \left| \int_t^\infty \lambda e^{-\lambda s} \left[ \int_E P(s, y, \Gamma) \nu(dy) - P(t, x, \Gamma) \right] ds \right| \\ &\leq \int_t^\infty \lambda e^{-\lambda s} \left| \int_E P(s, y, \Gamma) \nu(dy) - P(t, x, \Gamma) \right| ds \leq e^{-\lambda t}. \end{aligned}$$



Since  $\|\tilde{P}(t, x, \cdot) - q_\nu(\cdot)\|_{TV} = 2 \sup_{\Gamma \in \mathcal{E}} |\tilde{P}(t, x, \Gamma) - q_\nu(\Gamma)|$  [9, Appendix], it follows that  $\{\tilde{X}(t), t \in [0, \infty)\}$  is exponentially ergodic with index  $\lambda$  and  $M(x) = 2$ .

□

### 3 Moments and limits

In this section, we let the initial distribution of the original process be the Dirac measure concentrated at  $x \in E$ , where  $E = \mathbb{R}^n$ . Then we write  $\{\tilde{X}(t) : t \geq 0\}$  as  $\{(\tilde{X}_1(t), \dots, \tilde{X}_n(t)) : t \geq 0\}$ . Similar notations are introduced for the process  $\{X(t) : t \geq 0\}$ , too. Now consider the  $i$ th component process  $\{\tilde{X}_i(t), t \geq 0\}$ . From (1) we can obtain an expression for the moments

$$E_x[\tilde{X}_i^k(t)] = e^{-\lambda t} E_x[X_i^k(t)] + \int_E \int_0^t \lambda e^{-\lambda s} E_y[X_i^k(s)] ds \nu(dy), \quad (6)$$

where and below in this section, we assume the involved interchange of the order of integrals is legal, which is the case, for example, when

$$\int_E \int_0^t \lambda e^{-\lambda s} E_y[|X_i^k(s)|] ds \nu(dy) < \infty.$$

In turn, the equation (6) helps to establish the following bound.

**Proposition 1** *Let the  $k$ -th moment of the original process be exponentially bounded from the above in time, i.e.,*

$$E_x[X_i^k(t)] \leq c_{k,i}(x) e^{\eta_{k,i} t}, \quad (7)$$

where  $c_{k,i}(\cdot)$  is a measurable  $\nu$ -integrable function, and  $\eta_{k,i} < \lambda$  is a constant. Then,

$$\limsup_{t \rightarrow \infty} E_x[\tilde{X}_i^k(t)] \leq \frac{\bar{c}_{k,i} \lambda}{\lambda - \eta_{k,i}}, \quad (8)$$

where  $\bar{c}_{k,i} = \int_E c_{k,i}(y) \nu(dy)$ . If the  $k$ -th moment of the original process is uniformly bounded from the above (i.e.,  $E_x[X_i^k(t)] \leq c_{k,i} \in (-\infty, \infty)$ ), so is the  $k$ -th moment of the modified process by the same bound (i.e.,  $E_x[\tilde{X}_i^k(t)] \leq c_{k,i}$ ).

**Proof:** The equation (6) yields

$$\begin{aligned} E_x[\tilde{X}_i^k(t)] &\leq e^{-\lambda t} c_{k,i}(x) e^{\eta_{k,i} t} + \int_E \int_0^t \lambda e^{-\lambda s} c_{k,i}(y) e^{\eta_{k,i} s} ds \nu(dy) \\ &= e^{-(\lambda - \eta_{k,i})t} c_{k,i}(x) + \frac{\lambda}{\lambda - \eta_{k,i}} (1 - e^{-(\lambda - \eta_{k,i})t}) \int_E c_{k,i}(y) \nu(dy), \end{aligned}$$

which implies the first statement of the proposition. The second statement of the proposition also follows from the equation (6).

□

Now let us investigate what happens when the parameter  $\lambda$  goes to zero. Since  $\int_E P(s, y, \Gamma) \nu(dy)$  is a bounded measurable function with respect to  $s$ , according to [6], we can conclude that the limiting discounting is equivalent to the time averaging. This fact and Theorem 1 in [7] lead to the following statement.

**Theorem 3** *The existence of the limit  $\lim_{\lambda \rightarrow 0} q_\nu(\Gamma)$  is equivalent to the existence of the limit  $\lim_{T \rightarrow \infty} 1/T \int_0^T \int_E P(s, y, \Gamma) \nu(dy) ds$ . If these limits exist, they are equal. In particular, if the original process  $\{X(t) : t \in [0, \infty)\}$  is positive Harris recurrent, the limit  $\lim_{\lambda \rightarrow 0} q_\nu(\Gamma)$  exists and is equal to the invariant probability measure of the original process.*

**Remark 2** *There could be cases when the limit  $\lim_{\lambda \rightarrow 0} q_\nu(\cdot)$  exists for any probability distribution  $\nu(\cdot)$ . However, similarly to the case of singularly perturbed Markov processes [2], the original process might not be ergodic.*

## 4 Examples

Let us illustrate the general results with two examples.

### 4.1 Brownian motion with drift

As the first example let us consider the Brownian motion with drift  $\mu$  and variance coefficient  $\sigma^2$  on the real line  $E = \mathbb{R}$  (see e.g., [17]). The initial distribution of the original process is the Dirac measure concentrated at  $x \in E$ . It can be described in the stochastic differential notation

$$dX(t) = \mu dt + \sigma dW(t),$$

where  $W(t)$  is the standard Wiener process. The probability density function of the Brownian process has a closed form

$$p(t, 0, z) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(z - \mu t)^2}{2\sigma^2 t}\right).$$

Here we assume that the process starts from zero. We observe that  $p(t, 0, z)$  does not converge to a proper probability density as  $t$  goes to infinity. If the modified process restarts also from zero, by formula (1) we have

$$\begin{aligned} \tilde{p}(t, 0, z) &= \exp(-\lambda t) \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(z - \mu t)^2}{2\sigma^2 t}\right) \\ &+ \lambda \int_0^t \exp(-\lambda s) \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left(-\frac{(z - \mu s)^2}{2\sigma^2 s}\right) ds, \end{aligned} \tag{9}$$

which has the well defined limiting probability density function

$$q_0(z) = \lambda \int_0^\infty \exp(-\lambda s) \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left(-\frac{(z - \mu s)^2}{2\sigma^2 s}\right) ds.$$

Let us now consider that the original process starts with an arbitrarily fixed  $x \in E$  and restarts according to the distribution  $\nu(dy)$ , which admits a finite second moment. Then we may calculate the first moment of the modified process by formula (6) with  $E_x[X(t)] = x + \mu t$

$$\begin{aligned} E_x[\tilde{X}(t)] &= e^{-\lambda t}(x + \mu t) + \int_E \int_0^t \lambda e^{-\lambda s}(y + \mu s) ds \nu(dy) \\ &= e^{-\lambda t}(x + \mu t) + [1 - e^{-\lambda t}] \int_E y \nu(dy) + [1 - (1 + \lambda t)e^{-\lambda t}] \frac{\mu}{\lambda}. \end{aligned}$$

Thus,

$$E_x[\tilde{X}(t)] \rightarrow \int_E y \nu(dy) + \frac{\mu}{\lambda},$$

as  $t \rightarrow \infty$ .

Similarly, a direct calculation gives

$$E_x[\tilde{X}^2(t)] \rightarrow \frac{\sigma^2}{\lambda} + \frac{2\mu^2}{\lambda^2} + \int_E \left(\frac{2\mu y}{\lambda} + y^2\right) \nu(dy), \text{ as } t \rightarrow \infty,$$

and thus

$$Var_x[\tilde{X}^2(t)] \rightarrow \int_E y^2 \nu(dy) - \left(\int_E y \nu(dy)\right)^2 + \frac{\sigma^2}{\lambda} + \frac{\mu^2}{\lambda^2}, \text{ as } t \rightarrow \infty.$$

## 4.2 Geometric Brownian motion

As the second example we consider the geometric Brownian motion (see e.g., [17]), so that we take  $E = [0, \infty)$ . It can be described by the following stochastic differential equation (with the initial condition  $P(X(0) \in dy) = 1\{x \in dy\}$ , where  $x \in E$  is fixed)

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t),$$

where  $W(t)$  is the standard Wiener process. The probability density function of the geometric Brownian motion also has a closed form

$$p(t, x, z) = \frac{1}{\sqrt{2\pi}} \frac{1}{z\sigma\sqrt{t}} \exp\left(-\frac{(\ln(z) - \ln(x) - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t}\right), \quad (10)$$

which defines a log-normal distribution with mean

$$E_x[X(t)] = xe^{\mu t}, \quad (11)$$

and variance

$$\text{Var}_x[X(t)] = x^2 e^{2\mu t} (e^{\sigma^2 t} - 1). \quad (12)$$

Indeed, for any  $k = 1, 2, \dots$ , it holds for this log-normal distribution that

$$E_x[X^k(t)] = x^k \exp(k(\mu - \frac{\sigma^2}{2})t + \frac{k^2 \sigma^2 t}{2}).$$

Next, for any fixed  $k = 1, 2, \dots$ , we assume  $\lambda - k(\mu - \frac{\sigma^2}{2}) - \frac{1}{2}k^2 \sigma^2 \neq 0$ , and  $\nu(dy)$  has a finite  $k$ th moment. By using formula (6), we calculate the  $k$ th moment of the modified process

$$\begin{aligned} E_x[\tilde{X}^k(t)] &= e^{-\lambda t} x^k e^{k(\mu - \frac{\sigma^2}{2})t + \frac{1}{2}k^2 \sigma^2 t} + \int_E \int_0^t \lambda e^{-\lambda s} y^k e^{k(\mu - \frac{\sigma^2}{2})s + \frac{1}{2}k^2 \sigma^2 s} ds \nu(dy) \\ &= x^k e^{-t(\lambda - k(\mu - \frac{\sigma^2}{2}) - \frac{1}{2}k^2 \sigma^2)} + \int_E \lambda y^k \int_0^t e^{-s(\lambda - k(\mu - \frac{\sigma^2}{2}) - \frac{1}{2}k^2 \sigma^2)} ds \nu(dy) \\ &= x^k e^{-t(\lambda - k(\mu - \frac{\sigma^2}{2}) - \frac{1}{2}k^2 \sigma^2)} + \lambda \frac{1 - e^{-t(\lambda - k(\mu - \frac{\sigma^2}{2}) - \frac{1}{2}k^2 \sigma^2)}}{\lambda - k(\mu - \frac{\sigma^2}{2}) - \frac{1}{2}k^2 \sigma^2} \int_E y^k \nu(dy). \end{aligned}$$

Thus, when  $\lambda > k(\mu - \frac{\sigma^2}{2}) + \frac{1}{2}k^2 \sigma^2 > 0$ , we have

$$E_x[\tilde{X}^k(t)] \rightarrow \frac{\lambda}{\lambda - k(\mu - \frac{\sigma^2}{2}) - \frac{1}{2}k^2 \sigma^2} \int_E y^k \nu(dy)$$

as  $t \rightarrow \infty$ .

## References

- [1] H. Alt, L. Guibas, K. Mehlhorn, R. Karp and A. Wigderson, “A method for obtaining randomized algorithms with small tail probabilities”, *Algorithmica*, v.16, pp.543-547, 1996.
- [2] K.E. Avrachenkov, J. Filar and M. Haviv, “Singular Perturbations of Markov Chains and Decision Processes”, in *Handbook of Markov Decision Processes: Methods and Applications*, E.A. Feinberg and A. Schwartz (eds.), pp.113-150, 2002.
- [3] S. Brin and L. Page, “The anatomy of a large-scale hypertextual Web search engine”, *Computer Networks and ISDN Systems*, v.30, pp.107-117, 1998.
- [4] F. Chung, “The heat kernel as the pagerank of a graph”, *Proceedings of the National Academy of Sciences*, v.104, pp.19735-19740, 2007.
- [5] D. Down, S. Meyn and R. Tweedie, “Exponential and Uniform Ergodicity of Markov processes”, *Annals of Probability*, v.23, pp.1671-1691, 1995.
- [6] W. Feller, *An Introduction to Probability Theory and its Applications Vol. II*, 2nd ed., New York, John Wiley & Sons, 1971.

- [7] P. Glynn, “Some topics in regenerative steady-state simulation”, *Acta Applicandae Mathematicae*, v.34, pp. 225-236, 1994.
- [8] M.C. González, C.A. Hidalgo and A.-L. Barabási, “Understanding individual human mobility patterns”, *Nature*, v.453, pp.779-782, 2008.
- [9] O. Hernández-Lerma and J. Lasserre, *Discrete-time Markov control processes*, Springer-Verlag, NY, 1996.
- [10] I. Kontoyiannis and S.P. Meyn, “Spectral theory and limit theorems for geometrically ergodic Markov processes”, *The Annals of Applied Probability*, v.13, no.1, pp.304-362, 2003.
- [11] B. Krishnamurthy and J. Rexford, *Web Protocols and Practice: HTTP/1.1, Networking Protocols, Caching, and Traffic Measurement*, Addison Wesley, 2001.
- [12] S.E. Kuznetsov, “Any Markov process in a Borel space has a transition function”, *Theory of Probability and Its Applications*, v.25, no.2, pp.384-388, 1980.
- [13] M. Luby, A. Sinclair and D. Zuckerman, “Optimal speedup of Las Vegas algorithms”, *Information Processing Letters*, v.47, pp.173-180, 1993.
- [14] S.M. Maurer and B.A. Huberman, “Restart strategies and Internet congestion”, *Journal of Economic Dynamics and Control*, v.25, pp.641-654, 2001.
- [15] S. Meyn and R. Tweedie, “Stability of Markov processes III: Foster-Lyapunov criteria for continuous time processes”, *Advances in Applied Probability* v.25, pp. 518-548, 1993.
- [16] E. Nummelin, *General irreducible Markov chains and nonnegative operators*, Cambridge University Press, 1984.
- [17] S.M. Ross, *Stochastic Processes*, 2nd ed., Wiley, 1996.
- [18] W.R. Stevens, *TCP/IP Illustrated, Volume 1: The Protocols*, Addison Wesley, 1994.
- [19] P.D. Walsh, D. Boyer and M.C. Crofoot, “Monkey and cell-phone-user mobilities scale similarly”, *Nature Physics*, v.6, pp.929-930, 2010.

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2004 route des Lucioles - BP 93  
06902 Sophia Antipolis Cedex

Publisher  
Inria  
Domaine de Voluceau - Rocquencourt  
BP 105 - 78153 Le Chesnay Cedex  
[inria.fr](http://inria.fr)

ISSN 0249-6399